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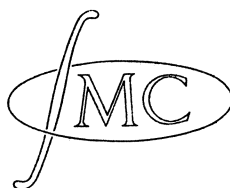
AFDELING MATHEMATISCHE STATISTIEK

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MUTUAL CHOICES

by

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Mutual Choices

1. Introduction

Let a group of n persons be given. Each person chooses k other persons ($1 \leq k \leq n-2$) at random, i.e. such that the $\binom{n-1}{k}$ ways in which a person can choose, have the common probability $\binom{n-1}{k}^{-1}$; the persons choose independently of each other. We will consider the problem of determining the distribution of the number of mutual choices.

It was a remark in Hofstätter's booklet [1] that led to the formulation of this problem.¹⁾ Hofstätter refers to his book [2] for the probabilistic background, but the treatment given there is limited to the derivation of the (binomial) distribution of the number of times that a given person is chosen.

It is immediately clear²⁾ that the expected value $\xi_{\underline{w}}$ of the number w of mutual choices is given by

$$\xi_{\underline{w}} = \binom{n}{2} \left(\frac{k}{n-1} \right)^2 = \frac{n k^2}{2(n-1)} \quad (1)$$

In the greater part of this preliminary report, our considerations will be limited to the case $k = 1$.

2. The distribution of $\underline{w}$ for $k = 1$

On determining the distribution of $\underline{w}$ for $k=1$, we will make use of some well-known identities, which we will

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2) Stochastic variables will be denoted by underlined letters.

recapitulate here. The notation is virtually the same as in Feller's book (Chapter IV in the second edition).

Let $A_1, \dots, A_N$ be an arbitrary collection of events, and let $\{i_1, \dots, i_r\}$ be any subset of $\{1, \dots, N\}$. We then define

$$p_{i_1, \dots, i_r} = P[A_{i_1}, \dots, A_{i_r}] \quad (2)$$

$$S_r = \sum p_{i_1, \dots, i_r} \quad (3)$$

where the summation is over all r - tuples from $\{1, \dots, N\}$. Then each of the quantities S_r and the probability Q_r that exactly r of the events $A_1, \dots, A_N$ are realized, can be expressed in the other, as follows:

$$Q_r = \sum_{j \geq r} (-1)^{j+r} \binom{j}{r} S_j \quad (4)$$

$$S_r = \sum_{i \geq r} \binom{i}{r} Q_i \quad (5)$$

In order to apply these identities, we define $A_{(i,j)}$ as the event "persons i and j choose each other". (Hence a "p with r indices" will become a "p with r pairs of indices"). The probability that two given persons choose each other is $(n-1)^{-2w}$, and therefore, because of the independency, the p's with w pairs of indices are equal to $(n-1)^{-2w}$ provided all $2w$ indices are different. Obviously, if not all indices are different, p will be zero. S_w will therefore be equal to $(n-1)^{-2}$ times the number of ways in which w disjoint pairs may be chosen from the set of all persons:

$$S_w = \frac{n!}{2^w w! (n-2w)! (n-1)^{2w}} \quad (6)$$

Formula (4) can now be applied to yield the following expression for the distribution of $\underline{w}$:

$$P[\underline{w} = w] = \frac{n!}{w!} \sum_{j \geq w} \frac{(-1)^{j+w}}{(j-w)! (n-2j)! 2^j (n-1)^{2j}} \quad (7)$$

3. The limiting distribution of $\underline{w}$ for arbitrary k .

For $k=1$, the limiting distribution of $\underline{w}$ when n tends to infinity can be found from (5) and (6) as follows. According to (5), the quantities $r! S_r$ are just the factorial moments of $\underline{w}$, whereas from (6) we have

$$\lim_{n \rightarrow \infty} r! S_r = \frac{1}{2^r}$$

Hence, the factorial moments of the limiting distribution are $\frac{1}{2^r}$, i.e. the limiting distribution is Poisson with mean $\frac{1}{2}$.

For $k > 1$ it is no longer true that all non-zero p 's with w indices are equal to each other. As a consequence, it is no longer feasible to give manageable formulae for S_w . (Even a seemingly simple case like $n=7$, $k=3$ leads to a total of about 150 different values of p 's). However, it is still possible to find the limiting distribution of $\underline{w}$, because for every value of $\underline{w}$, there is only one value of p that makes a significant contribution in the limiting case, as will be shown now.

Suppose a_j given persons are involved in exactly j mutual choices ($j=0, \dots, k$; $\sum j a_j = 2w$). In such a situation we have

$$p_{i_1 \dots i_w} = \prod_{j=0}^k \left\{ \frac{\binom{n-1-j}{k-j}}{\binom{n-1}{k}} \right\}^{a_j} = O(n^{-2w})$$

Consider the persons that are involved in at least one mutual choice. This group consists of $a_1 + \dots + a_k = n - a_0$ persons, who may be chosen in $\binom{n}{n-a_0}$ ways from the whole group. As $n - a_0 = a_1 + \dots + a_k \leq 2w$ is bounded, we have

$$\binom{n}{n-a_0} = O(n^{n-a_0})$$

Finally, the number of ways in which the above-mentioned $a_1 + \dots + a_k$ persons can allocate their mutual choices is independent of n , and hence S_w is the sum of a number of terms that are $O(n^{-2w+n-a_0})$, and not, as is easily verified $O(n^{-2w+n-a_0})$. The exponent is maximal when a_0 is minimal, i.e. when $a_0 = n - 2w$, $a_1 = 2w$, $a_2 = \dots = a_k = 0$, as is easily seen. Now the leading term in S_w can be determined in exactly the same way as was done for $k=1$, and we find

$$\lim_{n \rightarrow \infty} r! S_r = \left(\frac{k^2}{2}\right)^r \quad (8)$$

Therefore, also for $k > 1$ the limiting distribution of w is Poisson (with mean $\frac{k^2}{2}$)

4. The variance of w

For very low values of w it is still possible to determine S_w , for arbitrary values of k and n . For example, if one wants to calculate S_2 , only two cases have to be considered:

1. A and B choose each other, and C and D choose each other.
2. A and B choose each other, and A and C choose each other.

When S_2 is known, σ^2 can be found from the formula

$$\sigma^2 = 2S_2 + S_1 - S_1^2,$$

and after some calculations one finds

$$\sigma^2 = \frac{n k^2 (n-k-1)^2}{2(n-1)^3} \quad (9)$$

This expression remains unchanged when k is replaced by $n-k-1$, as is proper.

5. Extension to cycles of length greater than 2. ($k=1$)

Beside mutual choices, which can be interpreted as cycles of length 2, one might also consider cycles of length 3 (A chooses B, B chooses C, C chooses A) or greater. If $\underline{m}_i$ denotes the number of cycles of length i ¹⁾, the following relations can be proved.

$$\underline{m}_i = \frac{n!}{(n-i)! i (n-1)^i} \quad (10)$$

$$P[\underline{m}_i = m] = \frac{n!}{m!} \sum_{j \geq m} \frac{(-1)^{j+m}}{(j-m)! (n-ij)! \{i(n-1)\}^j} \quad (11)$$

$$\lim_{n \rightarrow \infty} \sum_{j \geq m} \frac{j!}{(j-m)!} P[\underline{m}_i = j] = \frac{1}{i^m} \quad (12)$$

From (12) it follows that the limiting distribution of the number of cycles of length i is Poisson with mean $\frac{1}{i}$ ($i \geq 2$). Hence, the number of persons involved in a cycle of length $i \geq 2$ is 1 on the average in the limiting case.

1) It will be convenient to have both $\underline{m}_2$ and $\underline{w}$ as symbols for the number of mutual choices.

6. Persons that do not belong to a cycle (k=1)

The expected value, ξ_{m_1} , of the number of persons that do not belong to a cycle, can immediately be found from (10):

$$\xi_{m_1} = n - \sum_{i=2}^n i \cdot \xi_{m_1=n-i} = n - \sum_{i=2}^n \frac{n!}{(n-i)! (n-1)!} i$$

This expression can also be written as follows

$$\xi_{m_1} = \frac{n^2}{n-1} - \frac{n!}{(n-1)^n} \psi(n-1) \quad (13)$$

where $\psi(n) = \sum_{j=0}^n \frac{n^j}{j!}$

It can be verified by partial integration that

$$\sum_{j=0}^n \frac{e^{-n} n^j}{j!} = 1 - \frac{1}{\Gamma(n+1)} \int_0^n t^n e^{-t} dt$$

The right hand is asymptotically $\frac{1}{2} + \frac{1}{3\sqrt{2\pi n}} + O(\frac{1}{n})$,

whence

$$\psi(n) \approx \left(\frac{1}{2} + \frac{1}{3\sqrt{2\pi n}} \right) e^n.$$

From (13) we then have

$$\boxed{\xi_{m_1} = n - \sqrt{\frac{\pi n}{2}} + O(1)} \quad (14)$$

To determine the distribution of m_1 , we need the following lemmata.

Lemma 1. If m is the number of persons that do not belong to a cycle, then these persons can choose in $\gamma(m, n)$ ways, where γ is given by

$$\gamma(m, n) = n^{m-m} \cdot n^{m-1}. \quad (15)$$

Proof: Let A be the set of all persons that do not belong to a cycle, C is the set of the persons that do belong to a cycle, and B the subset of A of persons that choose somebody in C. Suppose A is not empty. As no cycles occur in A, neither B nor C can be empty. Let b be the number of elements in B. Then there are $\binom{m}{b}$ possibilities for the set B, and the persons belonging to it can make their choices in $(n-m)^b$ ways. In order to determine the number of ways in which the m-b persons belonging to A-B can choose, we notice that this question is equivalent to the original problem, A-B playing the role of A, and B that of C. For, the persons belonging to A-B choose nobody from C (or else b would be larger), and they choose without cycles. Hence we may apply induction ¹⁾:

$$\gamma(m,n) = \sum_{b=1}^m \binom{m}{b} (n-m)^b \gamma(m-b,m) \quad (16)$$

Now, by substituting (15) into the right hand side of (16), the desired expression for $\gamma(m,n)$ reappears. We still have to show the correctness of (15) for $m=0$ and all n. This, however, is trivial.

Corrolary. Cayley's formula for the number of rooted trees that can be formed with m points, can be obtained from (15) as follows. The number is the same as the number of ways in which m persons can choose, without cycles, if there is exactly one "outward" choice, i.e. $m \cdot \gamma(m-1,m) = m^{m-1}$.

¹⁾ For this to be possible, it is also necessary that $m-b < m$ and $m < n$, which follows immediately from the non-emptiness of B and C.

Lemma 2. n persons who choose by cycles can do this in D_n ways, where D_n is n subfactorial ¹⁾

Proof Number the n persons $1, \dots, n$. It is easily seen that there is a one-to-one correspondence between patterns consisting of cycles only and permutations without invariant elements, and the lemma follows.

Using the lemmata, and the fact that the group of n can be divided in two groups of m and $n-m$ persons in $\binom{n}{m}$ ways, we find

$$P[\underline{m}_1 = m] = (n^m - m \cdot n^{m-1}) \binom{n}{m} D_{n-m} (n-1)^{-n}$$

or, equivalently

$$P[\underline{m}_1 = m] = \binom{n-1}{m} n^m D_{n-m} (n-1)^{-n} \quad (17)$$

As $\sum_{m=0}^{n-2} P[\underline{m}_1 = m] = 1$, we have the corollary

$$\sum_{m=0}^{n-2} \binom{n-1}{m} n^m D_{n-m} = (n-1)^n \quad (18)$$

The asymptotic behaviour of $\underline{m}_1$ may be determined as follows

$$P[\underline{m}_1 \leq m] = \sum_0^m \binom{n-1}{j} n^j D_{n-j} (n-1)^{-n}$$

If $n-j$ is large for all j between 0 and m , i.e. if $n-m$ is large, D_{n-j} can be approximated by $(n-j)! e^{-1}$. Substituting this, we find after some calculations

$$P[\underline{m}_1 \leq m] \approx \frac{(n-1)!}{e(n-1)^n} \frac{n^{m+1}}{m!}$$

¹⁾ A for our purpose, convenient way to define D_n is: the number of permutations of n elements $1, \dots, n$ such that element i is not in the i -th place, $i=1, \dots, n$.

For n and m large, Stirling's formula can be used:

$$P[\underline{m}_1 \leq m] \approx \left(\frac{n}{m}\right)^{m+\frac{1}{2}} e^{-(n-m)}.$$

By making the change of variable $\underline{m} = n - y\sqrt{n}$, we find

$$P[y \geq y] \approx \left(\frac{n}{n-y\sqrt{n}}\right)^{n-y\sqrt{n}+\frac{1}{2}} e^{-y\sqrt{n}} \approx e^{-y^2},$$

provided $y/\sqrt{n}$ tends to zero when n tends to infinity. To ensure that all approximations made are valid, it is sufficient to restrict the range of y by

$$n^{-\frac{1}{4}} \leq y \leq n^{-\frac{1}{4}}.$$

Indeed: $n-m=y\sqrt{n} \gg n^{\frac{1}{4}}$; $m=n-y\sqrt{n} \gg n-n^{\frac{3}{4}}$; $y/\sqrt{n} \leq n^{-\frac{1}{4}}$.

Finally, by making the change of variable $y^2 = z$, we conclude that the asymptotic distribution of $\frac{(n-\underline{m}_1)^2}{n}$ is exponential (with mean 1).

7. Persons that are not chosen

Katz [3] gives the following formula for the distribution of the number $\underline{r}_0$ of persons that are not chosen (arbitrary k)

$$P[\underline{r}_0=r] = \sum_{j=r}^{n-1-k} (-1)^{r+j} \binom{j}{r} \binom{n}{j} \binom{n-j}{k}^j \binom{n-j-1}{k}^{n-j} \binom{n-1}{k}^{-n}.$$

For $k=1$, $r=0$ this expression reduces to

$$P[\underline{r}_0=0] = \sum_{j=0}^{n-2} (-1)^j \binom{n}{j} (n-j)^j (n-j-1)^{n-j} (n-1)^{-n}.$$

A second formula for this probability can be found using (17). For, though $\underline{r}_0$ is generally less than $\underline{m}_1$, we still have

$$\underline{r}_0 = 0 \Leftrightarrow \underline{m}_1 = 0,$$

as is easily seen. Therefore,

$$P[\underline{r}_0=0] = P[\underline{m}_1=0] = D_n(n-1)^{-n},$$

and we have the identity

$$\sum_{j=0}^{n-2} (-1)^j \binom{n}{j} (n-j)^j (n-j-1)^{n-j} = D_n,$$

which can also be proved in more direct way by using $D_n = nD_{n-1} + (-1)^n$ and Abels' generalized binomial formula.

Katz & Powell [4] have also considered a more general case (where the number of choices made by the i -th person is a given number depending on i), and give some very complicated formulae for this case.

8. Some numerical results

Table 1 shows the distribution of the number of mutual choices for $k=1$ and some values of n . The distribution for $n=3,4,5,6,7$ was calculated with formula 20, section 9.

For $n=20$, formula (7) was used. Of course, $n=\infty$ corresponds to the Poisson distribution of the limiting case.

n ↓	$m_2 \rightarrow$				
	0	1	2	3	4
3	.250	.750	-	-	-
4	.370	.593	.037	-	-
5	.434	.508	.059	-	-
6	.471	.459	.069	.001	-
7	.495	.428	.074	.002	-
20	.574	.337	.079	.009	.001
∞	.607	.303	.076	.013	.002

Table 1. $P[\underline{m}_2=m_2]$, $k=1$.

Table 2 gives the distribution of the number of mutual choices for $n=5,6,7,\infty$, and $k=2$. The computations were done using formulae (2), (3) and (4). As we have said before, these computations are complicated by the fact that not all non-zero p 's are equal for $k > 1$. For low values of n , the distribution has a very small variance.

n ↓	$m_2 \rightarrow$							
	0	1	2	3	4	5	6	7
5	.003	.089	.401	.421	.085	.002	-	-
6	.014	.144	.386	.347	.101	.008	.000	-
7	.027	.176	.368	.308	.106	.014	.001	.000
∞	.135	.271	.271	.180	.090	.036	.012	.003

Table 2. $P[\underline{m}_2=m_2]$, $k=2$.

Table 3 shows the distribution of the number of persons who do not belong to a cycle, for $n=3,4,5,6,7$ and $k=1$. In spite of the fact that all rows in table 3 are increasing, the distribution has, for large values of n , its mode at approximately $n-\sqrt{n}$.

n ↓	$m_1 \rightarrow$					
	0	1	2	3	4	5
3	.250	.750	-	-	-	-
4	.111	.296	.593	-	-	-
5	.043	.176	.293	.488	-	-
6	.017	.084	.207	.276	.415	-
7	.007	.040	.116	.221	.257	.360

Table 3. $P[\underline{m}_1=m_1]$, $k=1$.

9. A different formula for the distribution of $\underline{w}$ ($k=1$)

Suppose n persons choose in such a way that m_i cycles of length i result ($i \geq 2$), and that m_1 persons do not belong to any cycle. Using lemma 1 (section 6) it is easily seen that the number of ways in which this can be done, is given by

$$f(y) = \frac{n! (n^{m_1} - m_1 \cdot n^{m_1-1})}{(1^{m_1} 2^{m_2} \dots) (m_1! m_2! \dots)}$$

Hence

$$P[\underline{w}=\underline{w}] = \sum_{\substack{y(n) \\ m_2=w}} f(y) \cdot (n-1)^{-n} \quad (19)$$

By collecting terms that correspond to the same value of m_1 , (19) can be reduced to

$$P[\underline{w}=\underline{w}] = \frac{n!}{(n-1)^{n_2} n_2^w w!} \sum_v \frac{(n-v)n^{v-1}}{v!} \frac{T(n-v-2w)}{(n-v-2w)!} \quad (20)$$

$$\text{where } T(j) = j! \sum^* \frac{1}{(3^{m_3} 4^{m_4} \dots) (m_3! m_4! \dots)} \quad (21)$$

The summation $\sum^*$ in (21) is over all partitions of j for which $m_1=m_2=0$.

According to (21), $T(j)$ can be interpreted as the number of permutations of j in which no cycles of lengths 1 or 2 occur. Applying the principle of inclusion and exclusion to the permutations of n without cycles of length 1, it can be shown that

$$T(n) = n! \sum_i \left(-\frac{1}{2}\right)^i \frac{D_{n-2i}}{i!(n-2i)!}$$

The first few non-trivial ¹⁾ values $T(n)$ are:
 $T(6) = 160$, $T(7) = 1140$, $T(8) = 8988$.

This approach, however, seems to be even less promising than the one chosen in section 2, when results for $k > 1$ are desired.

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¹⁾ If $n \leq 5$, $T(n) = D_n$, because the exclusion of the partitions with 1-cycles, or 2-cycles, or both, excludes all but the permutation consisting of one cycle of length n .

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